Conformal invariance = finiteness and beta deformed $\mathcal{N} = 4$ SYM theory

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ABSTRACT: We claim that if by a choice of the couplings the theory can be made conformally invariant (vanishing of the beta functions) it is automatically finite and vice versa. This is demonstrated by explicit example in supersymmetric gauge theory. The formalism is then applied to the beta deformed $\mathcal{N} = 4$ SYM theory and it is shown that the requirement of conformal invariance = finiteness can be achieved for complex parameter of deformations.

KEYWORDS: Renormalization Group, Supersymmetric gauge theory, AdS-CFT Correspondence, Duality in Gauge Field Theories.

Contents

1. Introduction

The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) attracts much attention these days providing the playground to test nonperturbative features of quantum field theory. This is related to the property of conformal invariance of SUSY theories which is unique for four dimensional quantum field theories [1]. $\mathcal{N} = 4$ SYM was the first example of this sort stimulating further progress [1]. Another remarkable feature of the $\mathcal{N} = 4$ SYM theory is that via the AdS/CFT correspondence [2] it is related to a supergravity theory and one can get deeper understanding of duality between these two theories. Combined information may lead to new insight in gauge theories beyond the usual PT.

Note that the above mentioned AdS/CFT correspondence requires from the field theory to be conformally invariant and not necessarily obtaining the full $\mathcal{N} = 4$ supersymmetry. From this point of view it would be interesting to consider the other conformally invariant theories and to find the corresponding supergravity backgrounds. Of special interest is a marginally deformed $\mathcal{N} = 4$ theory analyzed in [3] for which the supergravity dual description has been found in [4]. This the so-called β deformation of the original $\mathcal{N} = 4$ SYM theory has been studied in the context of AdS/CFT correspondence in [5 – 7] and [8]. The authors of [9, 10] studied the $\mathcal{N} = 4$ SYM theory with the aim to get the conditions for its finiteness and conformal invariance. They performed a thorough analysis of the UV divergences in the framework of dimensional regularization (reduction) and found out that one can reach the desired goal if the deformation parameter β is real. They also claim that the requirements of finiteness and conformal invariance are not simultaneously satisfied and if one requires only conformal invariance to be valid then the complex values of β are also allowed [10]. This problem has been also considered in [11] where it was shown that conformal invariance understood as vanishing of the beta function holds in all orders of PT for any complex value of the deformation parameter provided one properly adjusts the couplings.

The aim of this paper is to show that the above mentioned mismatch between conformal invariance and finiteness is a result of mistreatment of dimensional regularization (reduction). If applied properly, one can reach both conformal invariance and finiteness simultaneously, thus allowing for the complex β deformations. Moreover, one can construct the whole family of conformally invariant and finite $\mathcal{N} = 1$ SYM theories, however, their dual description is not known so far.

2. The general formalism

The problem of finiteness in SYM theories has been studied long time ago [12] and the formalism has been developed [13] that allows one to treat the theory within the dimensional regularization (reduction). For completeness we briefly summarize it below.

Let us consider an $\mathcal{N} = 1$ SYM theory formulated in terms of $\mathcal{N} = 1$ superfields with an arbitrary cubic superpotential containing some set of Yukawa couplings $\{y\}$. We assume that a theory is gauge invariant and for simplicity consider the background gauge. Then from the non-renormalization theorems [14] one gets that in the chiral sector only the propagators are divergent and have to be renormalized while the vertices are finite. As for the gauge sector, in background gauge the renormalization of the vertex coincides with that of the gauge propagator, so one can also consider the gauge propagator only [15].

At the one loop order to get the gauge propagator finite one has to make the proper choice of the matter superfields. The following requirement is to be satisfied [16]:

$$\sum_R T(R) = 3C_2(G), \tag{2.1}$$

where $T(R)$ is the Dynkin index of a given representation R and $C_2(G)$ is the quadratic Casimir operator of the group.

Provided the requirement (2.1) is satisfied the only divergence one should take care of is the one of the chiral field propagator. This is a consequence of the following theorem [17]:

Theorem: If $\mathcal{N} = 1$ supersymmetric gauge theory is finite in L loops, the gauge propagator is finite $L + 1$ loops.

The same statement follows also from the explicit expression for the gauge beta function written in terms of the anomalous dimensions of the chiral fields in some particular scheme [18]

$$\beta_g = g^2 \frac{\sum T(R) - 3C(G) - \sum T(R)\gamma(R)}{1 - 2gC(G)}, \quad g \equiv g^2/16\pi^2. \tag{2.2}$$

Thus, if the anomalous dimensions of the chiral fields vanish, so does the gauge and Yukawa beta functions and the theory is conformally invariant. In some other gauges (for instance in components) one can have non-vanishing anomalous dimensions of some fields or vertices, but the beta functions still vanish. This situation is also attributed to conformal invariance since only the gauge invariant quantities make sense. In what follows we will assume the simplest possibility when all anomalous dimensions vanish and will call this situation conformal invariance.

Now the question is: how to reach this goal, i.e. how to get conformal invariance? And the related one: is the theory finite (that is all divergences cancel) in this case? We show below how it may be done in the framework of dimensional regularization (reduction) and give a positive answer to the second question. First, we analyse both problems (conformal invariance and finiteness) separately and then show that this is the same.

To study conformal invariance or vanishing of the anomalous dimensions one has first to apply some regularization and some renormalization scheme. In general the anomalous dimensions are scheme dependent but if they vanish, they vanish in any scheme. We adopt dimensional regularization or more precisely dimensional reduction [19] since dimensional regularization does not support supersymmetry. We ignore the problem of inconsistency of dimensional reduction in higher orders [20] assuming that it is adjusted by finite corrections. We adopt also the $\overline{MS}$ renormalization scheme. Then the chiral field renormalization constant has the form

$$Z_{2i}^{-1} = 1 + \sum_{n=1}^{\infty} \frac{C_n^{(i)}(\{y\}, g)}{\varepsilon^n}, \quad C_n^{(i)}(\{y\}, g) = \sum_{k=n}^{\infty} C_{kn}^i(\{y\}, g), \tag{2.3}$$

where the coefficient functions $C_{kn}^i(\{y\}, g)$ are the homogeneous polynomials in y_i and g of the order of k.[1] The anomalous dimensions γ_i depend on renormalized couplings $\{y\}$ and g and are given by the single pole terms

$$\gamma_i(\{y\}, g) = \sum_k k C_{k1}^i(\{y\}, g). \tag{2.4}$$

In the lowest orders one has

$$\gamma_i(\{y\}, g) = B_{1j}^i y_j + B_{10}^i g + \sum_{j,k} B_{2jk}^i y_j y_k + \sum_j B_{2j}^i y_j g + B_{20}^i g^2 + \cdots., \tag{2.5}$$

where $B_{j\ldots}^i$ are some numbers.

The vanishing of anomalous dimensions can be achieved by choosing the Yukawa couplings in the form of perturbation series over g [12]

$$y_i = \alpha_{0i}^{(0)} g + \alpha_{1i}^{(0)} g^2 + \alpha_{2i}^{(0)} g^3 + \cdots, \tag{2.6}$$

where the coefficients $\alpha_{ni}^{(0)}$ are calculated order by order in PT solving the system of linear algebraic equations. To guarantee the existence of solution the one-loop matrix B_{1j}^i has to be non-degenerate. This has to be explicitly checked in one loop. Then the procedure works in all orders.

This is not enough, however, to cancel all the pole terms in Z factors (2.3). At the same time finiteness of Z would mean the finiteness of a theory. Indeed, to cancel the pole terms one has to write down eq. (2.6) for $\varepsilon \neq 0$ which means that one has a double

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[1]Hereafter for the shorthand notation we use $g = g^2/16\pi^2, y_i = y_i^2/16\pi^2$.

series [13]

$$
\begin{aligned}
y_i = {} & g\left(\alpha_{0i}^{(0)} + \alpha_{0i}^{(1)}\varepsilon + \alpha_{0i}^{(2)}\varepsilon^2 + \cdots + \alpha_{0i}^{(n-2)}\varepsilon^{n-2} + \alpha_{0i}^{(n-1)}\varepsilon^{n-1} + \alpha_{0i}^{(n)}\varepsilon^n + \cdots\right) \\
& + g^2\left(\alpha_{1i}^{(0)} + \alpha_{1i}^{(1)}\varepsilon + \alpha_{1i}^{(2)}\varepsilon^2 + \cdots + \alpha_{1i}^{(n-2)}\varepsilon^{n-2} + \alpha_{1i}^{(n-1)}\varepsilon^{n-1} + \cdots\right) \\
& + g^3\left(\alpha_{2i}^{(0)} + \alpha_{2i}^{(1)}\varepsilon + \alpha_{2i}^{(2)}\varepsilon^2 + \cdots + \alpha_{2i}^{(n-2)}\varepsilon^{n-2} + \cdots\right) \\
& + \cdots\cdots\cdots\cdots \\
& + g^{n-1}\left(\alpha_{n-2\,i}^{(0)} + \alpha_{n-2\,i}^{(1)}\varepsilon + \cdots\right) + g^n\left(\alpha_{n-1\,i}^{(0)} + \cdots\right).
\end{aligned}
\tag{2.7}
$$

In a given order of PT equal n one needs all terms of the double expansion with a total power of $g\cdot\varepsilon$ equal n. The existing freedom of choice of the coefficients $\alpha_{ki}^{(m)}$ is enough to get *simultaneously* the vanishing of the anomalous dimensions (read *conformal invariance*) and the pole terms in Z factors (read *finiteness*). The coefficients from $\alpha_{ni}^{(0)}$ to $\alpha_{0i}^{(n)}$ calculated in n-th order of PT are related. One can not put either of them to zero in an arbitrary way.

Notice, however, that if the renormalization constants Z_i are finite, there is no need to any renormalization at all. One can proceed with the unrenormalized expressions. To show this we again consider the chiral propagators. Consider the bare chiral propagator prior to any renormalization given by perturbative expansion (D-algebra had already been performed)

$$
\begin{aligned}
D_{iB}(\{y_B\}, g_B, p^2, \varepsilon) = {} & 1 + \sum_{n=1}^{\infty} \frac{1}{(p^2)^{n\varepsilon}}\left(\frac{d_n^i(y_B, g_B)}{\varepsilon^n} + \frac{d_{n-1}^i(y_B, g_B)}{\varepsilon^{n-1}} + \cdots \right. \\
& \left. + \frac{d_1^i(y_B, g_B)}{\varepsilon} + d_0^i(y_B, g_B) + \cdots\right).
\end{aligned}
\tag{2.8}
$$

The finiteness now means that all $d_n^i(y_B, g_B)$, $n > 0$ vanish. It is possible to achieve this goal without any preliminary renormalization in terms of the bare couplings. The bare couplings, contrary to the renormalized ones, do not depend on the renormalization scheme but on regularization. In case of a finite theory they are finite and related to the renormalized ones by finite renormalization which is scheme dependent.

The coefficient functions $d_n^i(y_B, g_B)$ are also the homogeneous polynomials over y_B and g_B and to achieve the vanishing of them one can choose the bare Yukawa couplings in the form of one fold ε expansion with positive powers of ε [13]

$$
y_{iB} = g_B(\alpha_{0i}^{(0)} + \alpha_{0i}^{(1)}\varepsilon + \alpha_{0i}^{(2)}\varepsilon^2 + \cdots).
\tag{2.9}
$$

The coefficients $\alpha_{0i}^{(n)}$ like $\alpha_{ni}^{(0)}$ above are calculated order by order of PT again solving the system of linear algebraic equations. In one loop this system of equations coincides with the one for determining the coefficients $\alpha_{ni}^{(0)}$ with modified r.h.s. and is solvable if the one loop matrix B_{1j}^i is not-degenerate. This requirement again guarantees the solution in all orders. Notice that the vanishing of the simple pole automatically leads to the vanishing of the higher order poles. This is the consequence of local renormalizability of quantum field theory.

One can see that the problem of finiteness is easier to address in terms of the bare quantities. Eq. (2.9) contrary to (2.7) is linear with respect to g_B and is easier to implement. But both the ways lead to the same statement: if the theory is finite it is conformally invariant and vice versa.

3. Example

To demonstrate how the above mentioned statements are explicitly realized in the framework of dimensional regularization (reduction) we consider a toy example which imitates the situation in beta deformed $\mathcal{N} = 4$ SYM theory.

Let us assume that we have a supersymmetric gauge theory with only one Yukawa coupling y corresponding to a triple interaction. Consider the propagator of a chiral superfield calculated up to three loops (D algebra had already been performed)

$$D_B(p^2, g_B, h_B) = 1 + \left(\frac{d_{11}}{\varepsilon} + d_{10} + d_{1-1}\varepsilon \right) \frac{1}{(p^2)^\varepsilon} + \left(\frac{d_{22}}{\varepsilon^2} + \frac{d_{21}}{\varepsilon} + d_{20} \right) \frac{1}{(p^2)^{2\varepsilon}}$$
$$+ \left(\frac{d_{33}}{\varepsilon^3} + \frac{d_{32}}{\varepsilon^2} + \frac{d_{31}}{\varepsilon} \right) \frac{1}{(p^2)^{3\varepsilon}} + \cdots, \tag{3.1}$$

where the coefficient functions $d_{ij} = d_{ij}(g_B, y_B)$ depend on the bare couplings and are the homogeneous polynomials of the order i.

The renormalization constant which makes the propagator finite in the $\overline{MS}$ scheme is

$$Z_2^{-1} = 1 + \frac{c_{11}}{\varepsilon} + \left(\frac{c_{22}}{\varepsilon^2} + \frac{c_{21}}{\varepsilon} \right) + \left(\frac{c_{33}}{\varepsilon^3} + \frac{c_{32}}{\varepsilon^2} + \frac{c_{31}}{\varepsilon} \right) + \cdots, \tag{3.2}$$

where the coefficients $c_{ij} = c_{ij}(g, y)$ depend on the renormalized couplings and are also the homogeneous polynomials of the order i. This expression allows one to define the anomalous dimension γ

$$\gamma(g, y) = c_{11} + 2c_{21} + 3c_{31} + \cdots \tag{3.3}$$

and the Yukawa beta function

$$\beta_y(g, y) = 3y\gamma(g, y). \tag{3.4}$$

The bare coupling y_B and the renormalized one are related by

$$y_B = yZ_2^{-3}, \tag{3.5}$$

where Z_2^{-1} is given by (3.2). Similarly for the gauge coupling one has

$$g_B = gZ_g, \tag{3.6}$$

where we define

$$Z_g = 1 + \frac{a_{11}}{\varepsilon} + \left(\frac{a_{22}}{\varepsilon^2} + \frac{a_{21}}{\varepsilon} \right) + \cdots, \tag{3.7}$$

and the gauge beta function is

$$\beta_g(g, y) = a_{11} + 2a_{21} + \cdots \tag{3.8}$$

For our purposes we will need it up to two loops.

Not all of these coefficients are independent. By pole equations [21] the coefficients of the higher order poles in Z factors can be expressed in terms of the single pole ones as

$$a_{22} = \frac{1}{2}\left[a_{11}a_{11} + a_{11}g\frac{da_{11}}{dg} + 3c_{11}y\frac{da_{11}}{dy}\right], \tag{3.9}$$

$$c_{22} = \frac{1}{2}\left[c_{11}c_{11} + a_{11}g\frac{dc_{11}}{dg} + 3c_{11}y\frac{dc_{11}}{dy}\right],$$

$$c_{33} = \frac{1}{3}\left[c_{11}c_{22} + a_{11}g\frac{dc_{22}}{dg} + 3c_{11}y\frac{dc_{22}}{dy}\right],$$

$$c_{32} = \frac{1}{3}\left[c_{11}c_{21} + 2c_{21}c_{11} + a_{11}g\frac{dc_{21}}{dg} + 2a_{21}g\frac{dc_{11}}{dg} + 3c_{11}y\frac{dc_{21}}{dy} + 6c_{21}y\frac{dc_{11}}{dy}\right].$$

Moreover, from the requirement that

$$Z_2^{-1}D_B(p^2, g_B, y_B) = \text{finite when } \varepsilon \to 0, \tag{3.10}$$

where for g_B and y_B one has to substitute expansions (3.5), (3.6), one finds the relations between the coefficients d_{ij} and c_{ij}. They are

$$d_{11} = -c_{11}, \tag{3.11}$$

$$d_{22} = c_{22},$$

$$d_{21} = -c_{21} - c_{11}d_{10} - a_{11}g\frac{dd_{10}}{dg} - 3c_{11}y\frac{dd_{10}}{dy},$$

$$d_{33} = -c_{33},$$

$$d_{32} = -c_{32} - c_{11}d_{21} - d_{11}c_{21} - d_{10}c_{22} - a_{11}g\frac{dd_{21}}{dg} - 3c_{11}y\frac{dd_{21}}{dy} - c_{11}a_{11}g\frac{dd_{10}}{dg},$$

$$-6c_{11}c_{11}y\frac{dd_{10}}{dy} - a_{22}g\frac{dd_{10}}{dg} - 3c_{22}y\frac{dd_{10}}{dy} - a_{21}g\frac{dd_{11}}{dg} - 3c_{21}y\frac{dd_{11}}{dy},$$

$$d_{31} = -c_{31} - c_{11}d_{20} - d_{10}c_{21} - d_{1-1}c_{22} - a_{11}g\frac{dd_{20}}{dg} - 3c_{11}y\frac{dd_{20}}{dy} - c_{11}a_{11}g\frac{dd_{1-1}}{dg},$$

$$-6c_{11}c_{11}y\frac{dd_{1-1}}{dy} - a_{21}g\frac{dd_{10}}{dg} - 3c_{21}y\frac{dd_{10}}{dy} - a_{22}g\frac{dd_{1-1}}{dg} - 3c_{22}y\frac{dd_{1-1}}{dy},$$

Having all these expressions one can demonstrate how the cancellation of divergences and nullification of the beta function work. To imitate the situation in beta deformed $\mathcal{N} = 4$ SYM theory we take the following expressions for the independent coefficient functions d_{ij} and a_{ij}

$$d_{11} = d_1(g - y), \tag{3.12}$$

$$d_{10} = d_0(g + y),$$

$$d_{1-1} = d_{-1}(g + y),$$

$$d_{21} = d_2(g^2 + gy + y^2),$$

$$d_{20} = d_{-2}(g^2 + gy + y^2),$$

$$d_{31} = d_3g^3 \text{ for } y = g,$$

$$a_{11} = 0,$$

$$a_{21} = a_2g(g - y).$$

The explicit form of d_{10}, d_{1-1}, d_{21} and d_{20} is not essential. What is important they do not vanish at $y = g$. For d_{31} we only need its value at $y = g$. Eq. (3.12) means that for the chiral propagator the UV divergence disappears for $y = g$ in the one loop order, but it does not disappear in two and three loops (in the real beta deformed $\mathcal{N} = 4$ SYM theory in the planar limit it disappears in 1, 2 and 3 loops [11] but does not disappear in 4 and 5 loops [9])). At the same time the gauge beta function identically vanishes in one loop and vanishes in two loops for $y = g$ (in the real beta deformed $\mathcal{N} = 4$ SYM theory it vanishes up to 4 loops for $y = g$).

Given eq. (3.12) one can find the remained coefficient functions. They are

$$a_{22} = 0, \tag{3.13}$$
$$c_{11} = d_1(y - g),$$
$$c_{21} = -d_2(g^2 + gy + y^2) - d_0d_1(y + g)(y - g) - 3d_0d_1y(y - g),$$
$$c_{22} = d_{22} = \frac{1}{2}d_1^2(y - g)(4y - g),$$
$$c_{31} = -d_3g^3 + 15d_0d_2g^3 \quad \text{for } y = g,$$
$$c_{32} = -2d_2d_1y(y^2 + yg + g^2) - d_2d_1(y - g)(3y^2 + 2yg + g^2) + 2/3a_2d_1g^2(y - g)$$
$$\qquad -d_1^2d_0(y - g)(20y^2 - 4yg - g^2),$$
$$d_{32} = -d_2d_1y(g^2 + gy + y^2) - d_2d_1(y - g)(5y^2 + 3gy + g^2) + 1/3a_2d_1g^2(y - g)$$
$$\qquad -d_1^2d_0(y - g)(10y^2 - 2gy - 1/2g^2),$$
$$c_{33} = -d_{33} = \frac{1}{6}d_1^3(y - g)(28y^2 - 20yg + g^2).$$

Now one can calculate the anomalous dimension γ according to eq. (3.3)

$$\gamma = d_1(y - g) - 2d_2(g^2 + yg + y^2) - 2d_0d_1(y - g)(4y + g) - 3d_3g^3 + 45d_0d_2g^3 + \cdots \tag{3.14}$$

Vanishing of γ can be achieved if one chooses the renormalized Yukawa coupling y in the form of perturbative expansion over g (see eq. 2.6))

$$y|_{\varepsilon=0} = g + \alpha_1^{(0)}g^2 + \alpha_2^{(0)}g^3 + \cdots \tag{3.15}$$

The requirement of vanishing of γ gives

$$\alpha_1^{(0)} = 6d_2/d_1, \quad \alpha_2^{(0)} = 3(d_3/d_1 + 12d_2^2/d_1^2 + 5d_0d_2/d_1).$$

So, one has

$$y|_{\varepsilon=0} = g + 6\frac{d_2}{d_1}g^2 + 3\left(\frac{d_3}{d_1} + 12\frac{d_2^2}{d_1^2} + 5\frac{d_0d_2}{d_1}\right)g^3 + \cdots \tag{3.16}$$

If eq. (3.16) is fulfilled then the anomalous dimension (and the beta function) vanishes up to three loops and one has conformal invariance. Since we claim that conformal invariance in this context is synonym to finiteness, let us check the cancellation of UV divergences. As was explained above we will need eq. (3.15) for $\varepsilon \neq 0$

$$y = g(1 + \alpha_0^{(1)}\varepsilon + \alpha_0^{(2)}\varepsilon^2 + \cdots) + g^2(\alpha_1^{(0)} + \alpha_1^{(1)}\varepsilon + \cdots) + g^3(\alpha_2^{(0)} + \cdots). \tag{3.17}$$

Notice that in the third order of PT the one should take into account all terms of the double expansion with the total power of $g \cdot \varepsilon$ equal 3.

Substituting eq. (3.17) into (3.2) one gets the remained coefficients

$$\alpha_0^{(1)} = -2d_2/d_1^2, \quad \alpha_0^{(2)} = \frac{2}{3d_1^2}\left(\frac{d_3}{d_1} + 6\frac{d_2^2}{d_1^2} - \frac{2a_2d_2}{3d_1^2} + 15\frac{d_0d_2}{d_1}\right),$$
$$\alpha_1^{(1)} = -\frac{2}{d_1}\left(\frac{d_3}{d_1} + 12\frac{d_2^2}{d_1^2} - \frac{2a_2d_2}{3d_1^2} + 15\frac{d_0d_2}{d_1}\right). \tag{3.18}$$

With this choice of coefficients all the pole terms in Z_2^{-1} cancel. Notice that if $\alpha_1^{(0)}$ is responsible for the cancellation of the two-loop anomalous dimension, both $\alpha_1^{(0)}$ and $\alpha_0^{(1)}$ are needed to cancel the $1/\varepsilon$ term in two loops. They also cancel the $1/\varepsilon^2$ term in three loops. Indeed, taking into account (3.17) it takes the form

$$\frac{1}{\varepsilon^2}: \ c_{32}|_{y=g} + y\frac{dc_{33}}{dy}|_{y=g}\alpha_0^{(1)} + y\frac{dc_{22}}{dy}|_{y=g}\alpha_1^{(0)} = \left[-6d_2d_1 + \frac{3}{2}d_1^3(-2)\frac{d_2}{d_1^2} + \frac{3}{2}d_1^2 6\frac{d_2}{d_1}\right]g^3 = 0.$$

Similarly, $\alpha_2^{(0)}$ is needed to cancel the three loop anomalous dimension and all three $\alpha_2^{(0)}$, $\alpha_1^{(1)}$ and $\alpha_0^{(2)}$ terms are used to cancel the $1/\varepsilon$ term in three loops.

Consider now the chiral propagator (3.1) and substitute our values of the coefficients d_{ij}. One has for the singular part

$$D_B(p^2, g_B, h_B) = 1 + \frac{d_1(g_B - y_B)}{\varepsilon}\frac{1}{(p^2)^\varepsilon} \tag{3.19}$$
$$+ \left(\frac{d_1^2(y_B - g_B)(4y_B - g_B)}{2\varepsilon^2} + \frac{d_2(g_B^2 + g_By_B + y_B^2)}{\varepsilon}\right)\frac{1}{(p^2)^{2\varepsilon}}$$
$$+ \left(\frac{-d_1^3(y_B - g_B)(28y_B^2 - 20y_Bg_B + g_B^2)}{6\varepsilon^3} + \frac{-d_2d_1y_B(y_B^2 + y_Bg_B + g_B^2)}{\varepsilon^2}\right.$$
$$\frac{-d_2d_1(y_B - g_B)(5y_B^2 + 3y_Bg_B + g_B^2) + 1/3a_2d_1g_B^2(y_B - g_B)}{\varepsilon^2}$$
$$\left.\frac{-d_1^2d_0(y_B - g_B)(10y_B^2 - 2y_Bg_B - 1/2g_B^2)}{\varepsilon^2} + \frac{d_3g_B^3}{\varepsilon}\right)\frac{1}{(p^2)^{3\varepsilon}}.$$

To get the cancellation of divergences in each order of perturbation theory one again has to choose the Yukawa coupling in a proper way in the form of ε expansion

$$y_B = g_B\left(1 + \alpha_0^{(1)}\varepsilon + \alpha_0^{(2)}\varepsilon^2 + \cdots\right). \tag{3.20}$$

Substituting this expansion into (3.19) and requiring the cancellation of divergencies one gets for $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ the same values as above (3.18). Contrary to the nullification of the anomalous dimension where the cancellation takes place between the lower and higher orders of PT, here the cancellation takes place within the same order between the higher and lower order pole terms. However, these two procedures are related since the higher order poles are given via RG pole equations by the lowest order expressions (see eq. (3.9)). Notice that the condition $y_B = g_B$ cancels the leading poles in all orders, the condition $y_B = g_B(1 + \alpha_0^{(1)}\varepsilon)$ cancels subleading poles in all orders, and the condition $y_B = g_B(1 +$

$\alpha_0^{(1)}\varepsilon + \alpha_0^{(2)}\varepsilon^2)$ cancels the subsubleading poles. In our case by the choice of $\alpha_0^{(1)}$ we cancel $1/\varepsilon$ term in two loops and simultaneously $1/\varepsilon^2$ term in three loops. The $\alpha_0^{(2)}$ term cancels the $1/\varepsilon$ term in three loops. So, one has

$$y_B = g_B\left(1 - 2\frac{d_2}{d_1^2}\varepsilon + \frac{2}{3d_1^2}\left(\frac{d_3}{d_1} + \frac{6d_2^2}{d_1^2} - \frac{2a_2d_2}{3d_1^2} + \frac{15d_0d_2}{d_1}\right)\varepsilon^2 + \cdots\right). \qquad (3.21)$$

If this conditions are satisfied then all divergences cancel and the theory is finite up to three loops. Further loops require new terms in eq. (3.21).

4. Beta deformed $\mathcal{N} = 4$ SYM theory in 4 loops

Consider now the beta deformed $\mathcal{N} = 4$ SYM theory. It is given by the action [9]

$$S = \int d^8z Tr\left(e^{-gV}\bar{\Phi}_ie^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z Tr(W^\alpha W_\alpha)$$
$$+ih\int d^6z Tr\left(q\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\Phi_1\Phi_3\Phi_2\right)$$
$$+i\bar{h}\int d^6\bar{z} Tr\left(\frac{1}{\bar{q}}\bar{\Phi}_1\bar{\Phi}_2\bar{\Phi}_3 - \bar{q}\bar{\Phi}_1\bar{\Phi}_3\bar{\Phi}_2\right), \quad q \equiv e^{i\pi\beta}, \qquad (4.1)$$

where the superfield strength tensor $W_\alpha = \bar{D}^2(e^{-gV}D_\alpha e^{gV})$ and Φ_i with $i = 1, 2, 3$ are the three chiral superfields of the original $\mathcal{N} = 4$ SYM theory in adjoint representation of the gauge group; h and β are complex numbers and g is the real gauge coupling constant. In the undeformed $\mathcal{N} = 4$ SYM theory one has $h = g$ and $q = 1$.

In the present case it is useful to define the couplings

$$h_1 \equiv hq, \quad h_2 \equiv h/q, \quad h_1^2 \equiv h_1\bar{h}_1, \quad h_2^2 \equiv h_2\bar{h}_2. \qquad (4.2)$$

The goal is to study the conditions that in the planar limit (large N of SU(N)) the couplings h_1^2 and h_2^2 have to satisfy in order to get conformal invariance of the theory for complex values of h and β. Explicit calculation gives the following values for the coefficient functions of the renormalization constant Z_2^{-1} in notation of the previous section [9] (For simplicity everywhere only the difference between the beta deformed and undeformed $\mathcal{N} = 4$ SYM theory is considered [11])

$$c_{nk} = F_{nk}(h_1^2, h_2^2, g^2) - (2g^2)^n, \quad n = 1\ldots3,\ k = 1\ldots3, \qquad (4.3)$$
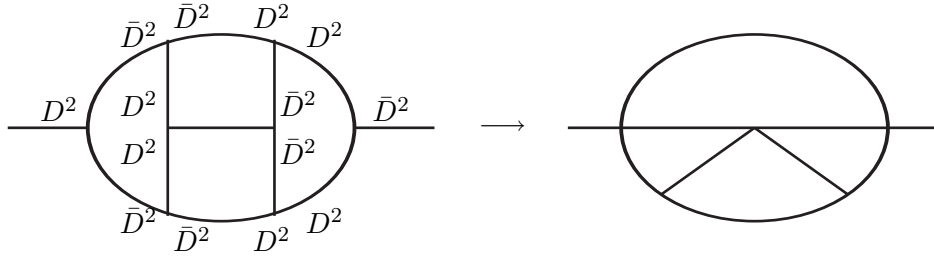
where the functions F_{nk} satisfy

$$F_{nk}(h_1^2 + h_2^2 = 2g^2) = (2g^2)^n.$$

Eq. (4.3) can be also rewritten as

$$c_{nk} = (h_1^2 + h_2^2 - 2g^2)P_{nk}(h_1^2, h_2^2, 2g^2), \quad n = 1\ldots3, k = 1\ldots3, \qquad (4.4)$$

where P_{nk} is a homogeneous polynomial of the order $n - 1$.

Figure 1: The only relevant divergent planar supergraph and its scalar counterpart at four loops

For $n = 4$ one has

$$c_{4i} = (h_1^2 + h_2^2 - 2g^2)P_{4i}(h_1^2, h_2^2, 2g^2), \quad i \neq 1 \tag{4.5}$$

$$c_{41} = (h_1^2 + h_2^2 - 2g^2)P_{41}(h_1^2, h_2^2, 2g^2) + G_{41}(h_1^2, h_2^2, 2g^2). \tag{4.6}$$

where $G_{41}(h_1^2, h_2^2, 2g^2)$ is a homogeneous polynomial of the fourth order that does not vanish at $h_1^2 + h_2^2 = 2g^2$. The latter contribution comes from the four loop chiral graph [9] (see figure 1). This graph has no divergent subgraphs and, therefore, has only primitive divergence.

Explicit form of c_{11} and c_{41} is

$$c_{11} = \left(-\frac{N}{(2\pi)^2} \right)(h_1^2 + h_2^2 - 2g^2) \doteq d_1(h_1^2 + h_2^2 - 2g^2) \tag{4.7}$$

$$c_{41} = \frac{5}{2}\zeta(5)\frac{N^4}{(2\pi)^8}[(h_1^2 + h_2^2)^4 - (2g^2)^4 + (h_1^2 - h_2^2)^4]$$

$$\doteq d_2[(h_1^2 + h_2^2)^4 - (2g^2)^4 + (h_1^2 - h_2^2)^4]. \tag{4.8}$$

Hereafter the chiral-gauge $\bar{\Phi}V\Phi$ contributions proportional to $h_1^2 + h_2^2 - 2g^2$ are omitted.

According to the recipe of the previous section one can now construct a conformal and finite theory choosing the renormalized couplings in the form of a double series of the fourth order

$$h_1^2 = g^2(a + \alpha_0^{(3)}\varepsilon^3) + g^4\alpha_1^{(2)}\varepsilon^2 + g^6\alpha_2^{(1)}\varepsilon + g^8\alpha_3^{(0)},$$

$$h_2^2 = g^2(b + \beta_0^{(3)}\varepsilon^3) + g^4\beta_1^{(2)}\varepsilon^2 + g^6\beta_2^{(1)}\varepsilon + g^8\beta_3^{(0)}. \tag{4.9}$$

Now, from the requirement of vanishing of anomalous dimension $\gamma = c_{11} + 2c_{21} + 3c_{31} + 4c_{41} = 0$, one finds

$$1 \; loop: \quad a + b = 2, \tag{4.10}$$

$$4 \; loops: \quad \alpha_3^{(0)} + \beta_3^{(0)} = \frac{-4\widehat{G}_{41}}{d_1 g^8} = \frac{-4(a-b)^4 d_2}{d_1},$$

where hereafter $\widehat{G}_{41}$ means that one has to take G_{41} at $h_1^2 + h_2^2 = 2g^2$.

To get $\alpha_0^{(3)}$ and $\beta_0^{(3)}$ one has to consider the bare propagator. Since the only nontrivial graph giving contribution to G_{41} has no divergent subgraphs the essential singular part of the bare propagator is

$$D_{41} = -G_{41}.$$

Therefore, the condition for its cancellation is

$$\widehat{P}_{44}g^2(\alpha_0^{(3)} + \beta_0^{(3)}) - \widehat{G}_{41} = 0.$$

This gives

$$\alpha_0^{(3)} + \beta_0^{(3)} = \frac{\widehat{G}_{41}}{\widehat{P}_{44}g^2}. \tag{4.11}$$

The value of $\widehat{P}_{44}$ can be calculated from the pole equations: $\widehat{P}_{44} = 9d_1^4g^6$, so that

$$\alpha_0^{(3)} + \beta_0^{(3)} = \frac{(a-b)^4 d_2}{9d_1^4}. \tag{4.12}$$

To reach total finiteness one can use the remaining parameters. From the requirement that $Z_2^{-1} = 1$ in four loops one gets

$$\widehat{G}_{41} + d_1 g^8(\alpha_3^{(0)} + \beta_3^{(0)}) + \widehat{P}_{22}g^6(\alpha_2^{(1)} + \beta_2^{(1)}) + \widehat{P}_{33}g^4(\alpha_1^{(2)} + \beta_1^{(2)}) + \widehat{P}_{44}g^2(\alpha_0^{(3)} + \beta_0^{(3)}) = 0. \tag{4.13}$$

This is one equation for two pairs of parameters. However, the same parameters are responsible for the cancellation of the second order pole in five loops. The fifth order coefficients are

$$c_{5i} = (h_1^2 + h_2^2 - 2g^2)P_{5i}(h_1^2, h_2^2, 2g^2), \qquad i = 3, 4, 5, \tag{4.14}$$
$$c_{5i} = (h_1^2 + h_2^2 - 2g^2)P_{5i}(h_1^2, h_2^2, 2g^2) + G_{5i}(h_1^2, h_2^2, 2g^2), \quad i = 1, 2$$

Having in mind expansion (4.9) the second order pole takes the form

$$\widehat{G}_{52} + \widehat{P}_{22}g^8(\alpha_3^{(0)} + \beta_3^{(0)}) + \widehat{P}_{33}g^6(\alpha_2^{(1)} + \beta_2^{(1)}) + \widehat{P}_{44}g^4(\alpha_1^{(2)} + \beta_1^{(2)}) + \widehat{P}_{55}g^2(\alpha_0^{(3)} + \beta_0^{(3)}) = 0. \tag{4.15}$$

The coefficient functions $\widehat{P}_{22}, \widehat{P}_{33}, \widehat{P}_{44}, \widehat{P}_{55}$ and $\widehat{G}_{52}$ can be found from the pole equations that gives

$$\widehat{P}_{22} = 3d_1^2g^2, \ \ \widehat{P}_{33} = 6d_1^3g^4, \ \ \widehat{P}_{44} = 9d_1^4g^6, \ \ \widehat{P}_{55} = \frac{54}{5}d_1^5g^8, \ \ \widehat{G}_{52} = \frac{24}{5}d_1\widehat{G}_{41}g^2.$$

Substituting these values into (4.13), (4.15) and taking into account eqs. (4.10), (4.12) one gets

$$\alpha_1^{(2)} + \beta_1^{(2)} = -\frac{2}{3}\frac{(a-b)^4 d_2}{d_1^3},$$
$$\alpha_2^{(1)} + \beta_2^{(1)} = 2\frac{(a-b)^4 d_2}{d_1^2}. \tag{4.16}$$

Provided eqs. (4.10), (4.12), (4.16) and (4.9) are satisfied one has totally consistent finite and conformally invariant theory (up to four loops) parameterized by two parameters a and b related by one condition $a+b = 2$. Apparently the mechanism will work in any loop order irrespectively of the explicit form of divergent terms. Looking back to the analysis of divergent structures in ref. [9] one finds that new chiral graphs always give contribution

proportional to $(h_1^2 - h_2^2)^4$, so that the compensating terms of expansion will be always proportional to $(a - b)^4$ as above.

In ref. [9, 10] it was claimed that the only reliable solution is $a = b = 1$. Otherwise one can not reach both the finiteness and conformal invariance simultaneously. We see that this statement is a result of mistreatment of dimensional regularization (reduction) in the process of cancellation of divergences: the authors of [9, 10] considered only the one fold expansion instead of two fold one (4.9). For the correct implementation of the procedure a is arbitrary and $b = 2 - a$. In fact, as one can see above, the requirement of cancellation of divergences always defines only the sum of α's and β's, thus allowing the whole family of solutions

$$h_1^2 + h_2^2 = \bar{h}h(\bar{q}q + 1/\bar{q}q) \tag{4.17}$$
$$= g^2 \left\{ 2 + \frac{5}{18}\zeta_5\delta^4\varepsilon^3 + \frac{5}{3}\zeta_5\delta^4\left(\frac{g^2 N}{4\pi^2}\right)\varepsilon^2 + 5\zeta_5\delta^4\left(\frac{g^2 N}{4\pi^2}\right)^2\varepsilon + 10\zeta_5\delta^4\left(\frac{g^2 N}{4\pi^2}\right)^3 + \cdots \right\},$$

where we denoted $a - b \equiv \delta$. For the bare couplings one has

$$h_1^2|_B + h_2^2|_B = g_B^2 \left\{ 2 + \frac{5}{18}\zeta_5\delta^4\varepsilon^3 + \cdots \right\}. \tag{4.18}$$

This permits, in particular, the value of $|q| \neq 1$, thus allowing one to obtain a complex deformation of the $\mathcal{N} = 4$ SYM theory with arbitrary complex β.

5. Conclusion

We conclude that properly treated β deformed $\mathcal{N} = 4$ SYM theory can be made *simultaneously* conformal invariant and finite since these two requirements are *identical*. This can be achieved by adjusting the Yukawa couplings order by order in PT. In the framework of dimensional regularization (reduction) this requires the double series over the gauge coupling g and the parameter of dimensional regularization ε. For the bare coupling, on the contrary, only the one fold series over ε is enough. The whole procedure depends on regularization (for bare quantities) and renormalization scheme (for the renormalized ones). In the other regularization techniques it looks differently but the main conclusion remains the same.

The analysed β deformed SYM theory represents the whole class of conformal $\mathcal{N} = 1$ SYM theories in four dimensions. They can be constructed by the same mechanism of adjustment of the corresponding Yukawa couplings. This adjustment has to be done order by order in PT. At the moment there is no any theory (except for $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SYM ones) for which the all loop solution is known. These theories may as well have a dual description in the framework of supergravities within the AdS/CFT correspondence and though the proper backgrounds are not found few steps in this direction have been made (see for example [22, 23]).

Acknowledgments

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